On the Midterm Examination

Data of the midterm exam:

mean 55.6

median 57.5

sd 19.2

max 94

min 11

Those who scored between 20 and 30 are in bad situation, but not hopeless. Need to work much harder. Handing in assignments surely helps. There will be more "standard questions" in the final examination.

Solutions to two questions in the midterm exam are provided.

1. Let f be a continuous function on $[0,\infty)$. Define a sequence of functions recursively by

$$f_n(x) = \int_0^x f_{n-1}(t)\,dt, \quad f_0 = f\ , n \geq 1\ .$$

Show that

$$f_n(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1} f(t)\,dt\ .$$

Solution. Use mathematical induction, $M(1)$ is straightforward. Assume $M(n)$ is valid. Let us prove $M(n+1)$. We have

$$\begin{aligned}
f_{n+1}(x) &= \int_0^x f_n(t)\,dt \\
&= \int_0^x \frac{1}{(n-1)!}\int_0^t (t-z)^{n-1} f(z)\,dzdt \quad \text{induction hypothesis} \\
&= \frac{1}{(n-1)!}\int_0^x \int_z^x (t-z)^{n-1} f(z)\,dtdz \quad \text{Fubini's theorem} \\
&= \frac{1}{(n-1)!}\int_0^x \frac{(x-z)^n}{n} f(z)\,dz \\
&= \frac{1}{n!}\int_0^x (x-t)^n f(t)\,dt\ .
\end{aligned}$$

2. Consider a function F defined in a region $D \subset \mathbb{R}^2$ which is nonnegative, continuous and vanishes at the boundary of D. Let $A(z)$ be the area of the set $E(z) = \{(x,y) \in D : F(x,y) \geq z\}$. Show that

$$\iint_D F(x,y)\,dA = -\int_0^\infty zA'(z)\,dz\ .$$

It is understood that $A(z) = 0$ when $E(z)$ is empty.

Solution. First of all, $A(z) = 0$ for all $z > M = \max F$. Therefore,

$$-\int_0^\infty zA'(z)dz = -\int_0^{M+1} zA'(z)\,dz = \int_0^{M+1} A(z)\,dz\ .$$

On the other hand, let $P = \{z_j\}$ be a partition of $[0, M+1]$. The integral $\iint_D F(x,y)\,dA$ can be well approximated by the Riemann sums

$$\sum_j A(z_j^*)\Delta z_j$$

where z_j^* is a tag point. Letting $\|P\| \to 0$, the Riemann sums tend to

$$\int_0^{M+1} A(z)\,dz \ .$$